Solution to Assignment 6

Section 7.2

18. If $f \equiv 0$, then result is trivial. Otherwise, since f is continuous on $[a, b]$, there exists $x_0 \in [a, b]$ s.t. $\sup f = f(x_0) > 0$. By continuity, for each small $\varepsilon > 0$, there is some δ such that $|f(x) - f(x_0)| < \varepsilon$, for all $x \in [x_0 - \delta, x_0 + \delta] \cap [a, b]$. Hence

$$\delta(f(x_0) - \varepsilon)^n < \int_{(x_0-\delta,\, x_0+\delta)\cap[a,b]} f^n \le \int_a^b f^n \le \int_a^b f(x_0)^n = f(x_0)^n(b-a)$$

$$\delta^{1/n}(f(x_0) - \varepsilon) < M_n = \left(\int_a^b f^n\right)^{1/n} \le f(x_0)(b-a)^{1/n}$$

Note that $\lim_{n\to\infty} a^{1/n} = 1 \ \forall\ a > 0$. Letting $n \to \infty$, by the squeeze theorem,

$$f(x_0) - \varepsilon \le \liminf_{n\to\infty} M_n \le \limsup_{n\to\infty} M_n \le f(x_0)$$

Letting $\varepsilon \to 0$, $\lim_{n\to\infty} M_n = f(x_0) = \sup\{f(x) : x \in [a, b]\}$.

19. Let P_n be the equal length partition of $[-a, 0], -a = x_0 < x_1 < \cdots < x_n = 0$, where $x_j = -a + ja/n,\ j = 0, \cdots, n$. Then we have

$$\int_{-a}^0 f = \lim_{n\to\infty} \sum_j f(x_j)\frac{a}{n} ,$$

see Theorem 2.6. On the other hand, $-x_j, j = 0, \cdots, n$, becomes a partition Q_n on $[0, a]$. Therefore,

$$\int_0^a f = \lim_{n\to\infty} \sum_j f(-x_j)\frac{a}{n} .$$

Using $f(-x) = f(x)$, we see that

$$\sum_j f(-x_j)\frac{a}{n} = \sum_j f(x_j)\frac{a}{n} ,$$

hence

$$\int_{-a}^0 f = \int_0^a f .$$

When f is odd, follow the same line but now using $\sum_j f(-x_j)\frac{a}{n} = -\sum_j f(x_j)\frac{a}{n}$ to get

$$\int_{-a}^0 f = -\int_0^a f .$$

Supplementary Exercise

1. Let $f_+(x) = \max\{f(x), 0\}$ and $f_-(x) = -\min\{f(x), 0\}$. Show that f_+ and f_- are both integrable when f is integrable on $[a, b]$.

   Solution. Use that relation $f_+(x) = \frac{1}{2}\big(|f(x)| + f(x)\big)$, and $f_-(x) = \frac{1}{2}\big(|f(x)| - f(x)\big)$ and the integrability of $|f|$, see Theorem 2.8(d).

2. Let g be differed from f by finitely many points. Show that g is integrable if f is integrable over $[a, b]$ and they have the same integral over $[a, b]$.

   Solution. For $\varepsilon > 0$, find a partition P so that

   $$\sum_P \mathrm{osc}_j f \Delta x_j < \varepsilon/2 \ .$$

   Let $a_1, \cdots, a_m$, be the points g and f differ. They belong to at most $2m$ many subintervals of P. Hence

   $$\sum_j \mathrm{osc}_j g \Delta x_j \leq \sum_P \mathrm{osc}_j f \Delta x_j + 2M \times 2m \times \|P\| \ .$$

   Now we can refine the length of P so small that $4Mm\|P\| < \varepsilon/2$. Then

   $$\sum_j \mathrm{osc}_j g \Delta x_j < \varepsilon/2 + \varepsilon/2 = \varepsilon \ ,$$

   so g is integrable. Now, let P_n with $\|P_n\| \to 0$ and choose tags equal to none of these a_j's. Then $S(g, \dot{P}_n) = S(f, \dot{P}_n)$, so their integrals are equal as $n \to \infty$.

   Alternate proof. Let $h = g - f$ so that h is equal to zero except at finitely many points. By Theorem 2.11, h is integrable and its integral is equal to 0. Therefore, $g = f + h$ is integrable and

   $$\int_a^b g = \int_a^b (f + h) = \int_a^b f + \int_a^b h = \int_a^b f \ .$$

3. Let f be non-negative and continuous on $[a, b]$. Show that $\int_a^b f = 0$ if and only if $f \equiv 0$.

   Solution. It suffices to show if f is not identically zero, then its integral is positive. Suppose there is some $x_0 \in [a, b]$ at which $f(x_0) = \alpha > 0$. By continuity, there is some small $\delta > 0$ such that $f(x) \geq \alpha/2$ for all $x \in I \equiv [x_0 - \delta, x_0 + \delta] \cap [a, b]$. Therefore,

   $$\int_a^b f \geq \int_I f \geq \int_I \frac{\alpha}{2} = \frac{\delta\alpha}{2} > 0 \ .$$

4. Order the rational numbers in $[0, 1]$ into a sequence $\{z_n\}$ and define

   $$\varphi(x) = \sum_{\{j,\ z_j < x\}} \frac{1}{2^j} \ .$$

   Show that φ is continuous at every irrational number but discontinuous at every rational number in $(0, 1)$.

Solution. Let x be rational. Then $x = z_k$ for some k. From the definition of ϕ we immediately obtain $\phi(z_k^+) - \phi(z_k^-) = 1/2^k$, so it has a jump at z_k. On the other hand, for $\varepsilon > 0$, we fix a large j_0 such that $\sum_{j=j_0}^{\infty} 2^{-j} < \varepsilon$. The finite points $z_1, \cdots, z_{j_0}$ are disjoint from x and we can find some $\delta > 0$ such that $(x-\delta, x+\delta)$ does not contain any $z_1, \cdots, z_{j_0}$. That is, $z_j \in (x-\delta, x+\delta)$ implies $j > j_0$. It follows that for $y \in (x-\delta, x+\delta), y > x$,

$$0 < \phi(y) - \phi(x) = \sum_{\{j:\ x \le z_j < y\}} \frac{1}{2^j} \le \sum_{j=j_0+1}^{\infty} \frac{1}{2^j} = \frac{1}{2^{j_0}} < \varepsilon \ .$$

Similarly, we have $0 < \phi(x) - \phi(y) < \varepsilon$ for $y \in [x-\delta, x)$.

Note. This function is strictly increasing. Since monotone functions are integrable, this example shows how complicated an integrable function could be. It has countably many discontinuity points spreading densely over the interval. Thomae's function is another example of the same nature, although it is not monotone.

5. Give two integrable functions f and Φ so that $\Phi \circ f$ is not integrable. Hint: Take f to be the Thomae's function.

   Solution. Take f to be the Thomae's function which has been shown to be integrable on $[0,1]$. Next consider $\Phi(x) = 0$ if $x = 0$ and $\Phi(x) = 1$ otherwise. Φ is bounded and has only one discontinuity point at $x = 0$ and hence integrable. However, the composite function $\Phi \circ f$ satisfies $\Phi \circ f(x) = 1, x \in \mathbb{Q}$ and $\Phi \circ f(x) = 0$ otherwise. It is not integrable, see Example 2.2.

6. Let $f \in \mathcal{R}[a,b]$ and $g \in C^1[c,d]$ where $f[a,b] \subset [c,d]$. Show that the composite $g \circ f \in \mathcal{R}[a,b]$. Here C^1 means continuously differentiable.

   Solution. By MVT,

   $$g(f(x)) - g(f(y)) = g'(c)(f(x) - f(y)) \ ,$$

   where c is between $f(x)$ and $f(y)$. By assumption g' is continuous here $|g'| \le M$ for some M. We have

   $$\sum_j \mathrm{osc}_j g \circ f \Delta x_i \le M \sum_j \mathrm{osc}_j f \Delta x_j \ ,$$

   and the desired conclusion comes from the second criterion.

   Note: As a consequence of this property, the functions $|f|, f^n\ (n \ge 1), e^f, \sin f$, etc, are all integrable when f is integrable.

7. (Optional). Let $f \in \mathcal{R}[a,b]$ and $g \in C[c,d]$ where $f[a,b] \subset [c,d]$. Show that the composite $g \circ f \in \mathcal{R}[a,b]$. Hint: For $\varepsilon > 0$, fix δ_0 such that $|g(z_1) - g(z_2)| < \varepsilon$ for $|z_1 - z_2| < \delta_0$. For $\varepsilon,\ \delta_0 > 0$, there exists a partition P such that $\sum_j osc_{I_j} f \Delta x_j < \varepsilon\delta_0$. Then apply the Second Criterion.

   Solution. Given $\varepsilon > 0$, we want to find a partition P such that

   $$\sum_j \mathrm{osc}_{I_j} \Phi(f(x)) \Delta x_j < \varepsilon \ .$$

   Indeed, letting $M = \sup |f|$, Φ is uniformly continuous on $[-M, M]$. Therefore, there exists some δ such that $|\Phi(z_1) - \Phi(z_2)| < \varepsilon$ whenever $|z_1 - z_2| < \delta, z_1, z_2 \in [-M, M]$. For $\varepsilon_1 = \varepsilon\delta > 0$, by the Second Criterion we can find a partition P on $[a,b]$ such that

   $$\sum_j \mathrm{osc}_{I_j} f \Delta x_j < \varepsilon_1 \ .$$

On any one of those subintervals over which osc$f$ is less than δ, we have osc $\Phi \circ f$ is less than ε. On the other hand,

$$\delta \sum_j {}'\Delta x_j \le \sum_j {}'\mathrm{osc}_{I_j} f \Delta x_j < \varepsilon_1 \ ,$$

where $\sum'$ denotes the summation over those subintervals the osc of f is greater than or equal to δ. Therefore,

$$\sum_j {}'\Delta x_j \le \frac{\varepsilon_1}{\delta} = \varepsilon \ .$$

Putting things together, we have

$$\sum_j \mathrm{osc}\ \Phi \circ f \Delta x_j = \sum_j {}'\mathrm{osc}\Phi \circ f \Delta x_j + \sum_j {}''\mathrm{osc}\Phi \circ f \Delta x_j \le C_1\varepsilon + (b-a)\varepsilon \ ,$$

where $\sum''$ denotes the summation over those subintervals where the osc of f is less than δ and C_1 is the oscillation of Φ over $[-M, M]$. Now we can adjust $(C_1 + (b-a))\varepsilon$ to ε.

Note: This result is more general than the previous one.

8. Let f be a continuous function on $[a, b]$ and g a nonnegative continuous function on the same interval. Prove the mean-value theorem for integral:

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx,$$

for some $c \in [a, b]$.

Solution. The case is trivial when $g \equiv 0$. So we assume that $g > 0$ somewhere so that its integral is positive over $[a, b]$. Then $\int_a^b g(x)\ dx > 0$. Let $M = \sup f,\ m = \inf f$. We have

$$m\int_a^b f \le \int_a^b fg \le M\int_a^b f \ ,$$

implies that

$$\int_a^b fg \Big/ \int_a^b g \in [m, M] \ .$$

As f is continuous, its range $f([a, b]) = [m, M]$. Therefore, there exists some $c \in [a, b]$ such that

$$f(c) = \int_a^b fg \Big/ \int_a^b g \ .$$

Note. Here we have used the fact that the image of an interval under a continuous function is again an interval. See Theorem 5.3.9 in [BS].

9. Evaluate the following limits:

(a)

$$\lim_{n\to\infty}\Big(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}\Big);$$

(b)

$$\lim_{n\to\infty}\frac{(n!)^{1/n}}{n}.$$

Solution. (a) We observe

$$\frac{1}{n+1}+\cdots+\frac{n}{n+n}=\frac{1}{n}\sum_{j=1}^{n}\frac{n}{n+j}=\sum_{j=1}^{n}\frac{1}{1+j/n}\ .$$

Using the integrability of the function $f(x)=1/(1+x)$, we see that

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\cdots+\frac{n}{n+n}\right)=\int_0^1\frac{1}{1+x}dx=\log(1+x)\Big|_0^1=\log 2\ .$$

(b) Taking log,

$$\frac{1}{n}\log n!-\log n=\frac{1}{n}\sum_{j=1}^{n}\big(\log j-\log n\big)=\frac{1}{n}\sum_{j=1}^{n}\log\frac{j}{n}\ .$$

Letting $g(x)=\log x,\ x\in[0,1]$,

$$\lim_{n\to\infty}\log\frac{(n!)^{1/n}}{n}=\lim_{n\to\infty}\frac{1}{n}\log n!-\log n=\lim_{j\to\infty}\frac{1}{n}\sum_{n=1}^{n}\log\frac{j}{n}=\int_0^1\log x dx=-1\ .$$

Hence

$$\lim_{n\to\infty}\frac{(n!)^{1/n}}{n}=e^{-1}\ .$$

10. (Optional)

(a) Establish the Cauchy-Schwarz Inequality in integral form: For integrable f and g on $[a,b]$,

$$\int_a^b|fg|\le\sqrt{\int_a^b f^2}\sqrt{\int_a^b g^2}.$$

(b) Deduce the following Cauchy-Schwarz Inequality for vectors

$$\sum_{k=1}^{n}|a_kb_k|\le\sqrt{\sum_{k=1}^{n}a_k^2}\sqrt{\sum_{k=1}^{n}b_k^2},$$

and equality holds.

Solution. Consider the expression

$$\int_a^b(f-tg)^2\ ,$$

which is non-negative for all $t\in\mathbb{R}$. It is equal to

$$p(t)\equiv\int_a^b f^2-2\left(\int_a^b f\int_a^b g\right)t+\left(\int_a^b g^2\right)t^2\equiv A+Bt+Ct^2\ ,$$

which is non-negative for all t. The discriminant of this quadratic polynomial must be non-positive, that is, $B^2-4AC\le 0$,the inequality holds.

11. (Optional.) Let J be a convex function on some $[-M, M]$ and $f \in R[0,1]$ satisfying $|f(x)| \leq M$. Establish Jensen's Inequality in integral form

$$J\left(\int_0^1 f(x)dx\right) \leq \int_0^1 J(f(x))dx \ .$$

Solution. Let P_n be the equal-length partition of $[0,1]$ into n many subintervals. Since $\sum_j \Delta x_j = 1$, we can regard each $S(f, \dot{P}_n) = \sum_j f(j/n)1/n$ as a convex combination, so

$$J\left(S(f, \dot{P}_n)\right) \leq \sum_j J(f(j/n))\frac{1}{n} \ .$$

Letting $n \to \infty$, by the continuity of J, we get the Jensen's Inequality as the limit.